Rate-and-state friction: From Analysis to Simulation

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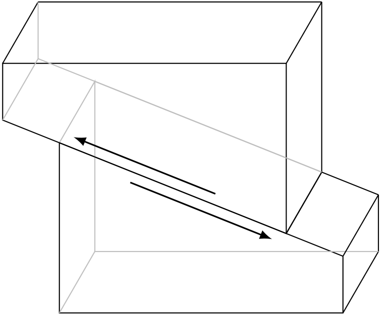
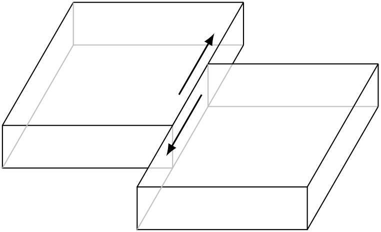
Types of faults: two examples



(a) A thrust fault[1]
Vertical, asymmetric arrangement



(b) A strike-slip fault:
Horizontal, symmetric arrangement

[1] Also known as a reverse dip-slip fault (as opposed to a normal fault).

Outline

Prime example: subduction zone

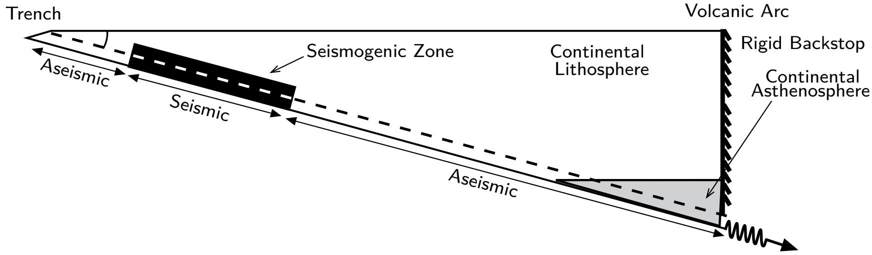


Figure: A subduction zone: the source of megathrust earthquakes

Modelling situation: bilateral contact; friction.

Simplifications: small deformation; small strain; one-body problem (bilateral contact with half-space); linear Kelvin–Voigt viscoelasticity.

Rate-and-state friction

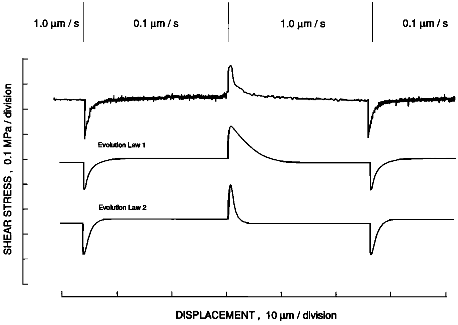


Figure: Velocity-stepping test; $|\boldsymbol{\sigma}_t| = \mu|\sigma_n| + C$, $\sigma_n = \text{const.}$ Measurements/ageing/slip law

Westerly granite inside a double direct shear apparatus
Source: M. F. Linker and J. H. Dieterich. "Effects of Variable Normal Stress on Rock Friction: Observations and Constitutive Equations". In: *Journal of Geophysical Research: Solid Earth* 97.B4 (1992), pp. 4923–4940. DOI: 10.1029/92JB00017

Clearly, we can write $\mu(t) = \mu(V)(t)$ but not $\mu(t) = \mu(V(t))$. Ruina's model takes the form

$$\mu(t) = \mu(V(t), \theta(t)) \quad \text{and} \quad \dot{\theta}(t) = g(\theta(t), V(t))$$

Restricted rate-and-state friction

We consider here only the case

$$\mu(t) = \mu(V(t), \alpha(t)) \quad \text{with} \quad \dot{\alpha}(t) + A(\alpha(t)) = f(V(t)).$$

with a monotone operator A and Lipschitz-continuous f.

Example: Dieterich's ageing law $\dot{\theta} = 1 - \frac{\theta V}{L}$ can be transformed to read

$$\dot{\alpha} - \frac{e^{-\alpha}}{\theta_0} = -\frac{V}{L}$$

with $\alpha = \log(\theta/\theta_0)$.

Not an example: Ruina's slip law $\dot{\theta} = -\frac{\theta V}{L} \log \frac{\theta V}{L}$ does not fit into this framework.

More on this matter in the appendix.

Strong-strong formulation

Our initial-value problem reads: find a displacement field $\boldsymbol{u}$ on Ω and a scalar state field α on the boundary segment Γ_C such that

$$
\begin{aligned}
\boldsymbol{\sigma} &= \boldsymbol{\mathcal{A}}\varepsilon(\dot{\boldsymbol{u}}) + \boldsymbol{\mathcal{B}}\varepsilon(\boldsymbol{u}) && \text{in } \Omega \times [0, T] \\
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} &= \rho \ddot{\boldsymbol{u}} && \text{in } \Omega \times [0, T] \\
\dot{\boldsymbol{u}} &= 0 && \text{on } \Gamma_D \times [0, T] \\
\boldsymbol{\sigma}\boldsymbol{n} &= 0 && \text{on } \Gamma_N \times [0, T] \\
\dot{\boldsymbol{u}} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_C \times [0, T] \\
\left.\begin{aligned} -\boldsymbol{\sigma}_t &= \frac{\mu(|\dot{\boldsymbol{u}}|, \alpha)|\bar{\sigma}_n| + C}{|\dot{\boldsymbol{u}}|}\dot{\boldsymbol{u}} \quad \text{for } \dot{\boldsymbol{u}} \neq 0 \\ |\boldsymbol{\sigma}_t| &\leq \mu(0, \alpha) + C \qquad\qquad \text{for } \dot{\boldsymbol{u}} = 0 \end{aligned}\right\} & && \text{on } \Gamma_C \times [0, T] \\
\dot{\alpha} + A(\alpha) &= f(|\dot{\boldsymbol{u}}|) && \text{on } \Gamma_C \times [0, T]
\end{aligned}
$$

Note that we replace the (unknown) σ_n with a fixed $\bar{\sigma}_n$. We also prescribe initial conditions on $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\ddot{\boldsymbol{u}}$, and α.

Weak-strong formulation

In a standard fashion we arrive at the weak formulation[2]

$$\boldsymbol{b}(t) \in \rho \ddot{\boldsymbol{u}}(t) + \mathfrak{A}\dot{\boldsymbol{u}}(t) + \mathfrak{B}\boldsymbol{u}(t) + \gamma^* \partial \Phi_\alpha(t,\cdot)(\gamma \dot{\boldsymbol{u}}(t))$$

with $\mathfrak{A}$, $\mathfrak{B}$ given by

$$\mathfrak{A}\boldsymbol{v} = \int_\Omega \langle \boldsymbol{\mathcal{A}}\varepsilon(\boldsymbol{v}), \varepsilon(\cdot)\rangle \quad \text{and} \quad \mathfrak{B}\boldsymbol{v} = \int_\Omega \langle \boldsymbol{\mathcal{B}}\varepsilon(\boldsymbol{v}), \varepsilon(\cdot)\rangle.$$

as well as the friction nonlinearities

$$\Phi_\alpha(t,\boldsymbol{v}) = \int_{\Gamma_C} \varphi_\alpha(t,x,|\boldsymbol{v}(x)|)\,\mathrm{d}x$$

$$\varphi_\alpha(t,x,v) = \int_0^v \mu(r,\alpha(t,x))|\bar{\sigma}_n| + C\,\mathrm{d}r.$$

We interpret A as a superposition operator in

$$\dot{\alpha}(t) + A(\alpha(t)) = f(|\gamma \dot{\boldsymbol{u}}(t)|).$$

[2]The precise solution spaces are mentioned on the next slide.

Analytical results

For the restricted type of rate-and-state friction (which makes additional assumptions on μ), we have been able to show:

- (2017) For any $T > 0$, we have a unique solution to the coupled weak-strong problem with

$$\boldsymbol{u} \in L^2(0, T, V), \quad \dot{\boldsymbol{u}} \in L^2(0, T, V), \quad \ddot{\boldsymbol{u}} \in L^2(0, T, V^*)$$
$$\alpha \in C(0, T, L^2(\Gamma_C))$$

  where $V = \{\boldsymbol{v} \in H^1(\Omega)^d \colon \boldsymbol{v} = 0 \text{ on } \Gamma_D,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_C\}$.
- (2014) For certain time-discretisation schemes (e.g. Newmark, backward Euler), one needs to solve problems of the form

$$\boldsymbol{b}_n \in \left(\frac{\lambda_M}{\tau}\rho + \mathfrak{A} + \frac{\tau}{\lambda_B}\mathfrak{B}\right)\dot{\boldsymbol{u}}_n(t) + \gamma^* \partial\Phi_{\alpha,n}(\gamma \dot{\boldsymbol{u}}_n)$$

  If each step is no larger in size than a certain constant, then all time steps have unique solutions $\boldsymbol{u}_n$, $\dot{\boldsymbol{u}}_n$, $\ddot{\boldsymbol{u}}_n \in V$ and $\alpha_n \in L^2(\Gamma_C)$.

In both cases, a fixed-point map is employed that turns into a contraction for sufficiently small time increments. The 2014 result thus also shows that a fixed-point iteration will converge regardless of the starting point.

Video

We run a simulation with the dimensions of (and parameters taken from) a lab-scale analogue model (more on that later).

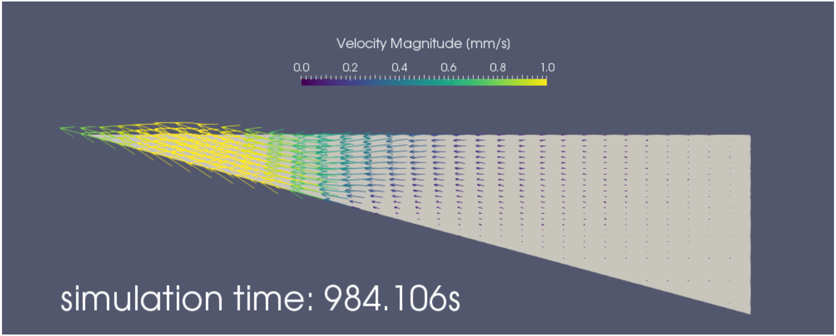


Figure: A still frame from the video

Spatial resolution

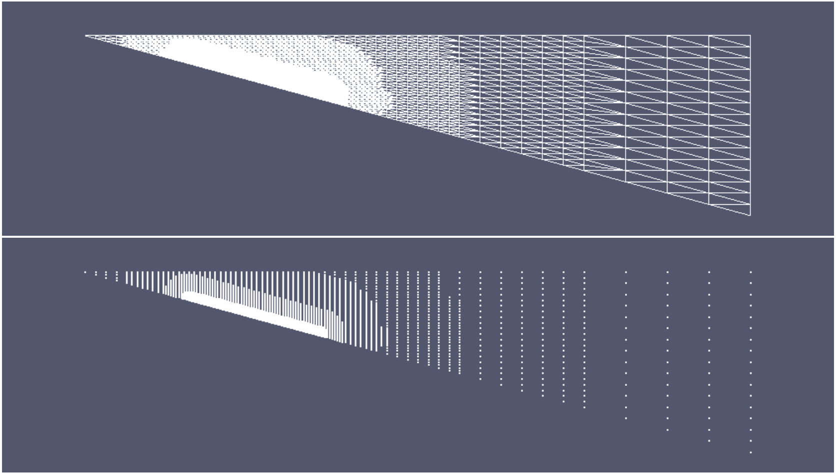


Figure: Actual spatial resolution of the simulation (wireframe / vertices as dots)

Surface uplift, numerical performance

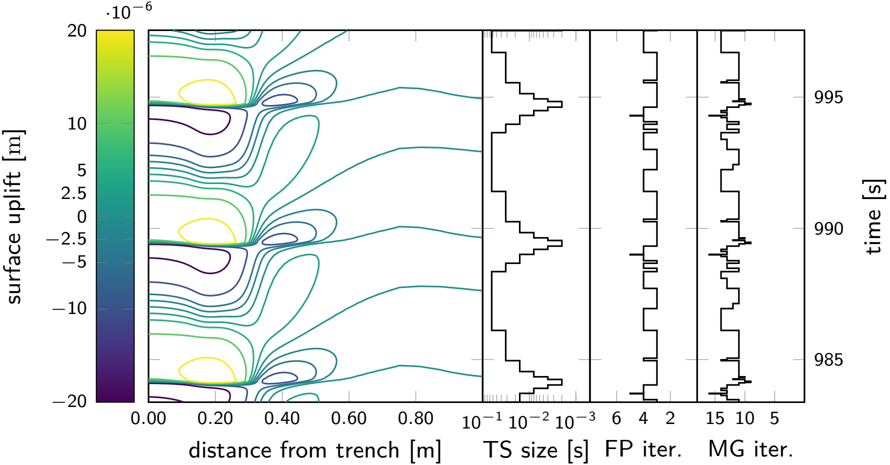


Figure: Surface uplift, performance of the numerical components:
(1) Adaptive time-stepping, (2) Fixed-point iteration, (3) TNNMG[3]

[3] Truncated Nonsmooth Newton Multigrid

Comparison: laboratory and experiment

Our simulation is based on the analogue model first presented here:

M. Rosenau, R. Nerlich, S. Brune, and O. Oncken. "Experimental insights into the scaling and variability of local tsunamis triggered by giant subduction megathrust earthquakes". In: Journal of Geophysical Research: Solid Earth 115.B9 (2010). DOI: 10.1029/2009JB007100

This allows us to compare our numerical results with lab measurements.

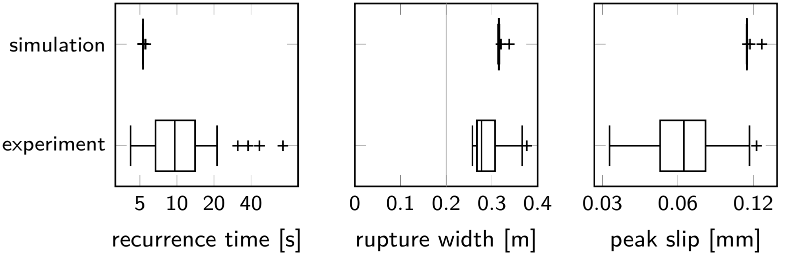


Figure: We isolate three key quantities.

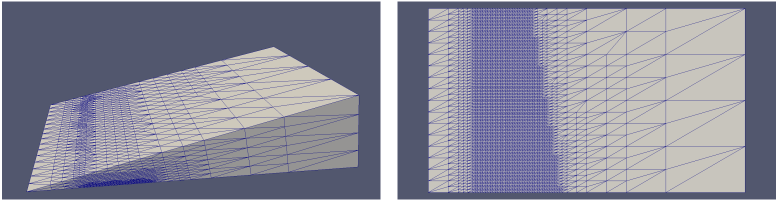


Figure: Computational 3D grid

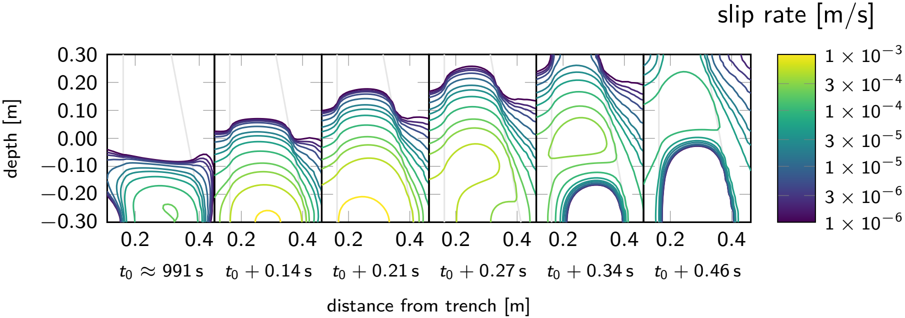
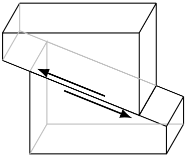


Figure: Coseismic evolution of sliding rate contours along seismogenic zone

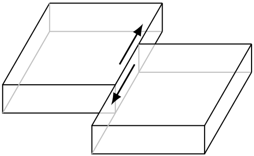
Reminder: one-body problem on thrust fault

Strong-strong problem: $\boldsymbol{u}$ on Ω, α on Γ_C.

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{B}\varepsilon(\boldsymbol{u}) && \text{in } \Omega \\
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} &= \rho \ddot{\boldsymbol{u}} && \text{in } \Omega \\
\dot{\boldsymbol{u}} &= 0 && \text{on } \Gamma_D \\
\boldsymbol{\sigma} \boldsymbol{n} &= 0 && \text{on } \Gamma_N \\
\dot{\boldsymbol{u}} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_C \\
-\boldsymbol{\sigma}_t &\in \partial \varphi_\alpha(t, \cdot)(|\dot{\boldsymbol{u}}|) && \text{on } \Gamma_C \\
\dot{\alpha} + A(\alpha) &= f(|\dot{\boldsymbol{u}}|) && \text{on } \Gamma_C
\end{aligned}$$



(a) Thrust fault



(b) Strike-slip fault

Attempt: two-body problem on strike-slip fault

Strong-strong problem: $\boldsymbol{u}^k$ on Ω^k, α^k on Γ_C^k; projections $\Psi^{i\to j}$ of Γ_C^i onto Γ_C^j.

$$
\begin{aligned}
\boldsymbol{\sigma}^k &= \boldsymbol{\mathcal{A}}\varepsilon(\dot{\boldsymbol{u}}^k) + \boldsymbol{\mathcal{B}}\varepsilon(\boldsymbol{u}^k) && \text{in } \Omega^k \\
\nabla \cdot \boldsymbol{\sigma}^k + \boldsymbol{b}^k &= \rho\ddot{\boldsymbol{u}}^k && \text{in } \Omega^k \\
\dot{\boldsymbol{u}}^k &= 0 && \text{on } \Gamma_D^k \\
\boldsymbol{\sigma}^k \boldsymbol{n} &= 0 && \text{on } \Gamma_N^k \\
(\dot{\boldsymbol{u}}^1 - \dot{\boldsymbol{u}}^2 \circ \Psi^{1\to 2}) \cdot \boldsymbol{n}^1 &= 0 && \text{on } \Gamma_C^1 \\
-\boldsymbol{\sigma}_t^1 &\in \partial\varphi_{\alpha}(t,\cdot)(\dot{\boldsymbol{u}}^1 - \dot{\boldsymbol{u}}^2 \circ \Psi^{1\to 2}) && \text{on } \Gamma_C^1 \\
\boldsymbol{\sigma}_t^1 &= -\boldsymbol{\sigma}_t^2 && \text{on } \Gamma_C^1 \\
\dot{\alpha}^1 + A(\alpha^1(t)) &= f(|\dot{\boldsymbol{u}}^1 - \dot{\boldsymbol{u}}^2 \circ \Psi^{1\to 2}|) && \text{on } \Gamma_C^1 \\
\dot{\alpha}^2 + A(\alpha^2(t)) &= f(|\dot{\boldsymbol{u}}^2 - \dot{\boldsymbol{u}}^1 \circ \Psi^{2\to 1}|) && \text{on } \Gamma_C^2
\end{aligned}
$$

Key questions: (1) Is there one state field or are there two? (2) If there are two, how do they enter into φ? If there is one, where does it live? (3) Both questions also arise for heterogeneous parameters.

Further reading

- E. Pipping. "Dynamic problems of rate-and-state friction in viscoelasticity". Dissertation. Freie Universität Berlin, 2014. URN: urn:nbn:de:kobv:188-fudissthesis000000098145-4.
- E. Pipping, R. Kornhuber, M. Rosenau, and O. Oncken. "On the efficient and reliable numerical solution of rate-and-state friction problems". In: *Geophysical Journal International* 204.3 (2016), pp. 1858–1866. DOI: 10.1093/gji/ggv512.
- E. Pipping. *Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction*. 2017. arXiv: 1703.04289v1.

History-dependent operators

Friction with a history-dependent operator R

$$\mu(t) = \mu(V(t), (RV)(t))$$

$$|(RV)(t) - (R\tilde{V})(t)| \leq L_R \int_0^t |V(s) - \tilde{V}(s)| \,\mathrm{d}s \qquad (1)$$

Example: state evolution equation with monotone A:

$$\mu(t) = \mu(V(t), \alpha(t)) \quad \text{with} \quad \dot{\alpha}(t) + A(\alpha(t)) = f(V(t)). \qquad (2)$$

If f is L_f-Lipschitz then (2) implies (1) with $L_R = L_f$.

Concrete example: ageing law $\dot{\theta} = 1 - \frac{\theta V}{L}$ can be transformed to read

$$\dot{\alpha} - \frac{e^{-\alpha}}{\theta_0} = -\frac{V}{L}$$

with $\alpha = \log(\theta/\theta_0)$.

Comparison of the friction frameworks

We have two formulations:

- rate-and-state:

$$\mu(t) = \mu(V(t), \theta(t)) \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}(\theta(t), V(t))$$

- history-dependent operators:

$$\mu(t) = \mu(V(t), (RV)(t)) \quad \text{with} \quad \|RV - R\tilde{V}\|_{L^\infty(0,t)} \leq L_R \|V - \tilde{V}\|_{L^1(0,t)}$$

Within the intersection of both models lies the restricted rate-and-state model with monotone A and Lipschitz-continuous f

$$\mu(t) = \mu(V(t), \alpha(t)) \quad \text{and} \quad \dot{\alpha}(t) + A(\alpha(t)) = f(V(t))$$

and thus in particular Dieterich's (transformed) ageing law.