Fixed points and iterations for nonexpansive maps

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Lipschitz mappings

Consider $T\colon C \to C$ with a closed, nonempty subset C of the Hilbert space H.

Definition

The map T is called Lipschitz (with constant q) if it satisfies

$$|Tx - Ty| \leq q|x - y| \quad \text{for every } x, y \in C.$$

Theorem (Banach 1922)

If T is q-Lipschitz with $q < 1$, then the following hold.

1. *The set of fixed points $F(T)$ is a singleton, i.e. $F(T) = \{x^*\}$.*
2. *For any $x_0 \in C$, the Picard iteration $T^n x_0$ converges to x^*.*

For $q \approx 1$, convergence will typically be very slow.

The nonexpansive case

Nonexpansive (i.e. 1-Lipschitz) maps typically do not have fixed points. They do if we require the following:

1. Convexity and e.g. boundedness of C.
2. Nice geometric structure of H.

Theorem (Maurey 1981; Dowling and Lennard 1997)

Let X be a subspace of L^1. Then it has the fixed point property

> *Every nonexpansive map $T\colon C \to C$ on every nonempty closed, convex, bounded set $C \subset X$ has a fixed point.*

if and only if X is reflexive.

Theorem (Kirk 1965)

If C is a nonempty, closed, convex, and bounded subset of a Hilbert space H and $T\colon C \to C$ is nonexpansive, then we have $F(T) \neq \emptyset$.

Applications

Linear problem $Ax = b$ with an symmetric positive semi-definite[1] square matrix A and $b \in \text{range}(A)$. Observe

$$Ax = b \iff Rx = 0 \iff x = (\text{id} + R)^{-1}x$$

with $Rx = Ax - b$. The resolvent $J = (\text{id} + R)^{-1}$ is

- defined everywhere ($\text{id} + R$ has strictly positive eigenvalues)
- nonexpansive

$$\begin{aligned}\langle Jx - Jy, Jx - Jy\rangle &\le \langle Jx - Jy, RJx - RJy\rangle + \langle Jx - Jy, Jx - Jy\rangle \\ &= \langle Jx - Jy, (id + R)Jx - (id + R)Jy\rangle \\ &= \langle Jx - Jy, x - y\rangle\end{aligned}$$

and thus $|Jx - Jy| \le |x - y|$ by Cauchy-Schwarz.

[1] $A^T = A$, $\langle Ax, x\rangle \ge 0$

Firm nonexpansiveness

The condition

$$\langle Jx - Jy, Jx - Jy \rangle \le \langle Jx - Jy, x - y \rangle$$

is called *firm nonexpansiveness*. It possesses a sense of direction and suggests that J is better behaved than a typical nonexpansive map.

Remark

There is a one-to-one correspondence between nonexpansive and firmly nonexpansive operators on H through the transformation

$$T \mapsto \frac{\mathrm{id} + T}{2} \quad \text{and its inverse} \quad T \mapsto 2T - \mathrm{id}\,.$$

Observe: $|u|^2 \le \langle u, w \rangle \iff \underbrace{\left|\tfrac{1}{2}w\right|^2 - \langle u, w \rangle + |u|^2}_{\left|\frac{1}{2}w - u\right|^2} \le \left|\tfrac{1}{2}w\right|^2.$

Iterations

Picard iterations $T^n x$ of nonexpansive operators typically do not converge, not even weakly.[2]

There are mainly two popular alternative iteration schemes.

- (Mann 1953; Krasnoselski 1955):

$$x_{n+1} := \alpha x_n + (1-\alpha) T x_n.$$

  Motivation: Picard iteration for a nicer map.

- (Halpern 1967):

$$x_{n+1} := \alpha_n x_0 + (1-\alpha_n) T x_n.$$

  Motivation: If C is closed, convex, and bounded, then the fixed points x_λ of $x \mapsto \lambda x_0 + (1-\lambda) T x$ converge strongly to $P_{F(T)}(x_0)$ as $\lambda \to 0$ (Browder 1967).

Many extensions exist (Ishikawa 1974; B. Xu and Noor 2002; Kim and H.-K. Xu 2005; Temir 2010).

[2] Think of a rotation.

Convergence Analysis: Mann's method

Theorem (Opial 1967; Edelstein and O'Brien 1978)

If C is closed and convex (not necessarily bounded) and $T : C \to C$ has a fixed point, then we have $x_n \xrightarrow{\sigma} x^* \in F(T)$ as $n \to \infty$ for

$$x_{n+1} := \alpha x_n + (1 - \alpha) T x_n .$$

whenever $\alpha \in (0, 1)$. The point x^* depends on x_0.

Counterexample (Genel and Lindenstrauss 1975)

At least for $\alpha = 1/2$, convergence is not strong.

Convergence rate for a rotation with $\alpha = 1/2$ and $|x_0| = 1$.

angle [°]	iterations	error	angle [°]	iterations	error
20	753	1×10^{-5}	5	12 091	1×10^{-5}
10	3020	1×10^{-5}	11.3384	2349	1×10^{-5}

Convergence Analysis: Halpern's method

Theorem (Wittmann 1992)

If C is closed and convex (not necessarily bounded) and $T: C \to C$ has a fixed point, then we have $x_n \to P_{F(T)}(x_0)$ as $n \to \infty$ for

$$x_{n+1} \coloneqq \alpha_n x_0 + (1 - \alpha_n) T x_n.$$

whenever $\alpha_n \to 0$, $\sum \alpha_n = \infty$, and $\sum |\alpha_n - \alpha_{n+1}| < \infty$.

The first two conditions are also generally necessary (Halpern '67).
General convergence result (Cominetti, Soto, and Vaisman 2014):

$$|Tx_n - x_n| \leq \operatorname{diam}(C) \Big(\pi \underbrace{\sum \alpha_i (1 - \alpha_i)}_{\approx \log(n) + \gamma - \pi^2/6 \text{ for } \alpha_i = 1/(i+1)}\Big)^{-1/2}$$

Convergence rate for a rotation with $\alpha_n = 1/(n+1)$ and $|x_0| = 1$.

angle [°]	iterations	error	angle [°]	iterations	error
20	17	1×10^{-16}	5	71	1×10^{-16}
10	35	2×10^{-16}	11.3384	9683	1×10^{-5}

Detour: Haugazeau's hybrid method

Theorem (Brègman 1965)

For k-many closed convex bodies C_i with $S = \bigcap_{i \leq k} C_i \neq \emptyset$, the iteration of cyclic projections

$$x_{n+1} := P_{C_{(n \bmod k)+1}}(x_n)$$

converges weakly to a point $x^* \in S$. The point x^* depends on x_0.

Theorem (Haugazeau 1968)

We have strong convergence to $P_S(x_0)$ for the Q-stabilised iteration

$$x_{n+1} := Q(x_0, x_n, P_{C_{(n \bmod k)+1}}(x_n)),$$

with $Q(x, y, z) = P_{H(x,y) \cap H(y,z)}(x)$ and

$$H(u, v) = \{w\colon (w - v, v - u) \geq 0\}$$

Observe: The projectors P_{C_i} are firmly nonexpansive.

A general weak-to-strong principle

For cyclic projections, firmly nonexpansive operators, etc..

Theorem (Bauschke and Combettes 2001)

If $T\colon H \to H$ is firmly nonexpansive and the set of fixed points $F(T)$ is nonempty, then the iteration given by

$$x_{n+1} := Q(x_0, x_n, Tx_n) = P_{H(x_0,x_n)\cap H(x_n,Tx_n)}(x_0)$$

converges strongly to $P_{F(T)}(x_0)$.

For a firmly nonexpansive operator T and $y \in F(T)$ we have

$$0 \leq \langle (\mathrm{id}-T)x - (\mathrm{id}-T)y, Tx - Ty\rangle = \langle x - Tx, Tx - y\rangle,$$

thus $y \in H(x, Tx)$ and $F(T) \subset \bigcap_{x\in H} H(x, Tx)$. Conversely: $z \in H(z, Tz)$ implies $z \in F(T)$.

angle [°]	iterations	error	angle [°]	iterations	error
20	9	9×10^{-16}	5	36	4×10^{-15}
10	18	2×10^{-15}	11.3384	16	3×10^{-15}

Summary

Pro: None of the shortcomings of the other methods
Neither: Additional projection step very cheap (see next slide).
Contra: Rate of convergence unknown.

Further remarks

- Weak-to-strong principle has more general applications.
- Works also for finite families of firmly nonexpansive maps, ensuring convergence towards the projection onto the set of common fixed points.
- For infinite families, some assumptions have to be made; regardless the strategy can be applied to $(\mathrm{id} + \gamma_n R)^{-1}$ with $\inf_n \gamma_n > 0$
- More generally, the strategy can be applied to the proximal point method (Rockafellar 1976)
- Any iteration with a firmly nonexpansive map is a special case of the proximal point algorithm (Eckstein et al. 1988), e.g. Douglas-Rachford (Lions and Mercier 1979).

Appendix

The map Q can be explicitly calculated (Haugazeau 1968). To that end, define

$$\tilde{Q}(x,y,z) = \begin{cases} z & \text{if } \rho = 0 \text{ and } \chi \geq 0 \\ x + \left(1 + \frac{\chi}{\nu}\right)(z - y) & \text{if } \rho > 0 \text{ and } \chi\nu \geq \rho \\ y + \frac{\nu}{\rho}\Big(\chi(x-y) + \mu(z-y)\Big) & \text{if } \rho > 0 \text{ and } \chi\nu < \rho \end{cases}$$

where $\chi = \langle x - y, y - z \rangle$, $\mu = |x - y|^2$, $\nu = |y - z|^2$, and $\rho = \mu\nu - \chi^2$.

We now have the following dichotomy.

- Either $\rho = 0$ and $\chi < 0$, so that $H(x,y) \cap H(y,z) = \emptyset$ or
- the intersection $H(x,y) \cap H(y,z)$ is nonempty and we have $P_{H(x,y) \cap H(y,z)}(x) = Q(x,y,z) = \tilde{Q}(x,y,z)$.

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