Variational Methods for Rate- and State-dependent Friction

Elias Pipping[1]

1) Mathematisches Institut, Freie Universität Berlin, Germany, pipping@math.fu-berlin.de

Extended abstract

We consider a fault zone between two layers of rock. Relative to one layer, we can view the other as an elastic body that slides along a rigid surface subject to rate- and state-dependent friction given by the law

$$\mu = \mu_0 + a \log(V/V_0) + b \log(V_0\theta/L)$$

where μ is the coefficient of friction, V is the velocity, a, b, and L are material constants, V_0 and μ_0 are a reference velocity and friction coefficient, respectively, and θ is a state variable. The state is assumed to evolve either according to Dieterich's law $\dot{\theta} = -\theta V/L \log(\theta V/L)$ (also known as aging or slowness law) or Ruina's law $\dot{\theta} = 1 - \theta V/L$ (also known as slip law).

This type of friction is well-established and has been applied to a wide range of materials[2].

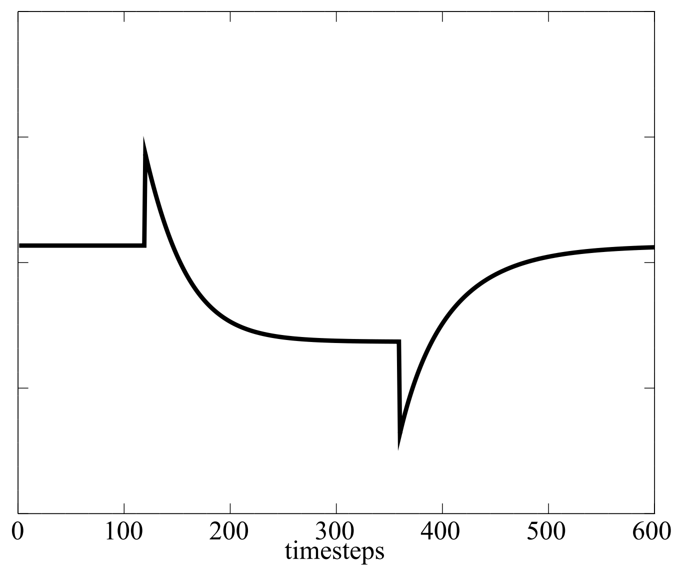

Figure 1: The friction coefficient under simulated velocity stepping

The above law can be motivated through so-called velocity stepping tests: A body is made to slide at a velocity v_1 for some time (120 timesteps in this case, see also figure 1), then instantaneously accelerated to a velocity v_2 at which it remains (for 240 timesteps in this case) until it is — again instantaneously — decelerated back to the velocity v_1. The coefficient of friction appears to be subject to a direct rate effect which leads to immediate jumps, the magnitude of which is given by the constant a, as well as a state-dependent effect which acts over time and whose magnitude is given by b.

The motion of the aforementioned body is assumed to be quasistatic, i.e. of negligible acceleration. We can now formulate a problem, discretise it in time, and find it to be a coupling of two variational subproblems – one of state and another of displacement – both of which can be shown to possess unique solutions. We repose those problems in terms of Finite Elements.

The state problem comprises an application of the backward Euler scheme to an ordinary first-order differential equation on the contact surface of the body. For the displacement, we construct a non-smooth but convex energy functional, which attains its minimum only at the solution. The corresponding (unregularised) problem of energy minimisation can be handled using the Truncated Nonsmooth Nonlinear Multigrid method (TNNMG)[1]. This method consists of at least two solvers, which it alternates between: First we have e.g. a nonlinear Gauss-Seidel smoother, which minimises the energy in each coordinate direction, one after another. Its benefit is its unconditional albeit potentially slow convergence. If we picture an approximation of the solution covered with errors of different frequencies, it becomes clear that such a solver will only address those of high-frequency. The second solver is a multigrid solver, which acts on a hierarchy of grids of varying coarseness, thereby adressing a wide range of frequencies. On each grid, it linearises the energy whenever possible, seeking solutions using Newton's method. This second solver is what the TNNMG method owes its speed to.

To find a solution for the overall problem, we alternate between the two subproblems.

The resulting algorithm has been implemented on top of the DUNE (`http://www.dune-project.org`) framework, allowing it to handle 2D and 3D problems; figure 1 is the result of such a computation.

References

[1] Carsten Gräser, Uli Sack, and Oliver Sander, *Truncated nonsmooth Newton multigrid methods for convex minimization problems*, Domain decomposition methods in science and engineering XVIII, Lect. Notes Comput. Sci. Eng., vol. 70, Springer, Berlin, 2009, pp. 129–136. MR 2743965

[2] Chris Marone, *Laboratory-derived friction laws and their application to seismic faulting*, Annual Review of Earth and Planetary Sciences **26** (1998), no. 1, 643–696.